View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

JOURNAL OF ALGEBRA 34, 430–443 (1975)

On the Existence of Automorphism Free Steiner Triple Systems

CHARLES C. LINDNER*

Mathematics Department, Auburn University, Auburn, Alabama 36830

AND

ALEXANDER ROSA†

Mathematics Department, McMaster University, Hamilton, Ontario L8S 4K1, Canada

Communicated by Marshall Hall, Jr.

Received November 5, 1973

1. INTRODUCTION

A Steiner triple system (briefly STS) is a pair $(S, \mathscr{B})$ where S is a set and $\mathscr{B}$ is a collection of 3-subsets of S (called triples) such that every 2-subset of S is contained in exactly one triple of $\mathscr{B}$. The number $|S|$ is called the order of the STS $(S, \mathscr{B})$. It is well known that there is an STS of order v if and only if $v \equiv 1$ or $3 \pmod 6$. Therefore in saying that a certain property concerning STS is true for all v it is understood that $v \equiv 1$ or $3 \pmod 6$. An STS of order v will sometimes be denoted by $STS(v)$.

An isomorphism from $(S_1, \mathscr{B}_1)$ onto $(S_2, \mathscr{B}_2)$ is a bijection $\alpha: S_1 \to S_2$ such that $\mathscr{B}_1\alpha = \mathscr{B}_2$. An automorphism of $(S, \mathscr{B})$ is an isomorphism of $(S, \mathscr{B})$ onto itself. Clearly the automorphisms of $(S, \mathscr{B})$ form a group $\Gamma(S, \mathscr{B})$ under composition of mappings. It is an open problem [5] whether, given an abstract group G, there always exist an STS whose automorphism group is isomorphic to G. It is believed that an important step towards settling this problem (in the affirmative) may be provided by determining those orders v for which there exists an $STS(v)$ admitting only the trivial automorphism. Such STS will be called *automorphism free* (briefly AF STS).

The purpose of this paper is to show that an AF $STS(v)$ exists if and only

* Research supported by NSF Grant GP-37629.

† Research supported by NRC Grant No. A7268.

Copyright © 1975 by Academic Press, Inc.
All rights of reproduction in any form reserved.

if $v \geqslant 15$ and that the number $R(v)$ of nonisomorphic $AF\ STS(v)$ goes to infinity with v. This result is proved in Section 3 via recursive combinatorial constructions. In Section 4 we give a table listing the best results to date for $R(v)$ from $v = 15$ to 103. Many of the bounds given in this table were obtained by techniques other than those developed in this paper. These constructions will appear in a subsequent paper by the authors.

2. Preliminaries

Given a graph G we denote by $V(G)$ the vertex-set of G and by $E(G)$ the edge-set of G (for undefined graph-theoretical notions see [8]). If the complete graph K_{2n} has vertex-set $T = \{\infty, 0, 1,..., 2n-2\}$ then $GK_{2n} = \{G_1, G_2,..., G_{2n-1}\}$, where $G_i = \{[\infty, i] \cup [i-j, i+j] \mid j = 1, 2,..., n-1\}$, and i, $i-j$, $i+j$ are taken modulo $2n-1$, is a 1-factorization of K_{2n} [8]. The series GK_{2n} of 1-factorizations and its properties have been studied by several authors [1, 2, 9, 15]. Wallis [15] has determined the structure of the 2-factor obtained as a union of two 1-factors of GK_{2n}. In particular, $G_i \cup G_j$ is a Hamiltonian circuit of K_{2n} if and only if $i-j$ and the modulus $2n-1$ are relatively prime; i.e., $(i-j, 2n-1) = 1$. Anderson [2] has determined $\Gamma(GK_{2n})$, the automorphism group of GK_{2n} (considered as a permutation group acting on $V(K_{2n})$) when $2n-1$ is a prime. In this case, $|\Gamma(GK_{2n})| = (2n-1)(2n-2)$ provided $n \geqslant 4$. It is not difficult to deduce from the results of [2] and [15] that $|\Gamma(GK_{2n})| \leqslant (2n-1)(2n-2)$ for arbitrary $n \geqslant 4$.

A 1-factorization $H = \{H_1, H_2,..., H_{2s-1}\}$ of K_{2s} is said to be a sub-1-factorization of a 1-factorization $F = \{F_1, F_2,..., F_{2n-1}\}$ of K_{2n} provided $V(K_{2s}) \subseteq V(K_{2n})$ and for each $i = 1, 2,..., 2s-1$, one can find a subscript $j(i)$ such that $E(H_i) \subseteq E(F_{j(i)})$; the number n/s is said to be the index of H in F.

We need one more auxiliary device. The following definitions are taken from [12].

An (A, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2,..., 2k\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2,..., k$. Similarly, a (B, k)-system is a set of k disjoint pairs (p_r, q_r) covering the elements of $\{1, 2,..., 2k-1, 2k+1\}$ exactly once and such that $q_r - p_r = r$ for $r = 1, 2,..., k$. It is known (see, e.g., [12]) that an (A, k)-system exists if and only if $k \equiv 0$ or $1 \pmod 4$, and a (B, k)-system exists if and only if $k \equiv 2$ or $3 \pmod 4$. Let us remark that an (A, k)-system and a (B, k)-system is essentially the same thing as what is called in [14] a Skolem $(2, k)$-sequence and a hooked Skolem $(2, k)$-sequence, respectively.

3. The Spectrum of AF STS

The structure and automorphism groups of all STS of order $v \leqslant 15$ has been determined for some time. It is well-known that there are no AF $STS(v)$ for $v \leqslant 13$. Therefore whenever we assume the existence of an AF $STS(v)$ it is understood that $v \geqslant 15$.

THEOREM 3.1. *If there exists an* AF $STS(v)$ *then there exists an* AF $STS(2v+1)$.

Proof. Let $(S, \mathscr{B})$ be an AF $STS(v)$, $S = \{a_1, a_2, ..., a_v\}$. Put $v + 1 = 2n$ and consider the 1-factorization $GK_{2n} = \{G_1, ..., G_{2n-1}\}$ of K_{2n} on $T = \{\infty, 0, 1, ..., 2n-2\}$. Put $S^* = S \cup T$, $\mathscr{B}^* = \mathscr{B} \cup \mathscr{C}$ where $\mathscr{C} = \{\{a_i, x, y\} \mid [x, y] \in G_i\}$. Obviously $(S^*, \mathscr{B}^*)$ is an $STS(2v+1)$. Let us show that the only automorphism of $(S^*, \mathscr{B}^*)$ is the identity.

I. Assume first that there is a non-trivial automorphism $\sigma \in \Gamma(S^*, \mathscr{B}^*)$ which maps $(S, \mathscr{B})$ onto itself. Then, since $(S, \mathscr{B})$ is an AF STS, $a_i\sigma = a_i$ for all $i = 1, 2, ..., v$, and each factor G_i must be mapped onto itself by σ. On the other hand, no element of T can be fixed by σ (since $t\sigma = t$ for some $t \in T$ implies $t\sigma = t$ for all $t \in T$). Therefore an element $i \in T$ must be mapped by σ onto a distinct element $j \in T$. But in order that the factor G_k containing the edge $[i, j]$ be mapped onto itself by σ, we must have $j\sigma = i$. It follows that σ is a product of n disjoint 2-cycles. Consider two cases:

Case 1. $n \equiv 1 \pmod 2$. Since there are $v = 2n - 1$ factors G_i and only n distinct 2-cycles of σ, there is a factor G_k containing no edge whose end-vertices are contained in the same 2-cycle of σ. But then G_k cannot be mapped onto itself by σ.

Case 2. $n \equiv 0 \pmod 2$. Without loss of generality we may assume $(0, \infty)$ to be a 2-cycle of σ. Then for any factor containing edges $[0, i]$ and $[\infty, j]$ there must be another factor containing edges $[0, j]$ and $[\infty, i]$. This implies $i \equiv 2j$, $j \equiv 2i \pmod v$ which obviously cannot be satisfied for every i (this is satisfied by exactly one pair i, j provided $v \equiv 3 \pmod{12}$).

II. Assume now that there is a non-trivial automorphism $\sigma \in \Gamma(S^*, \mathscr{B}^*)$ which maps $(S, \mathscr{B})$ onto $(S', \mathscr{B}')$ where $(S', \mathscr{B}')$ is another $STS(v)$. Then, by [10], $(S \cap S', \mathscr{B} \cap \mathscr{B}')$ is an $STS(\frac{1}{2}(v-1))$. We may assume without loss of generality that $S \cap S' = \{a_1, a_2, ..., a_{(1/2)(v-1)}\}$. Let $\mathscr{B}'' = \mathscr{B}' \backslash \mathscr{B}$ and let $H_i = \{[x, y] \mid \{a_i, x, y\} \in \mathscr{B}''\}$. A simple numerical argument then shows that each H_i contains the same number $\frac{1}{2}n$ of edges and $H = \{H_1, H_2, ..., H_{(1/2)(v-1)}\}$ is a 1-factorization of some K_n with $V(K_n) \subseteq T$. That is, H is a sub-1-factorization of G with index 2. It follows that for

$i \neq j$, $i, j \in \{1, 2,..., \frac{1}{2}(v-1)\}$, the union $G_i \cup G_j$ cannot be a Hamiltonian circuit of K_{2n}. On the other hand, no matter how we select $\frac{1}{2}(v-1)$ indices from the set $\{1, 2,..., v\}$ obviously there will always be a pair of indices, say i, j whose difference will be 1 or 2 and therefore relatively prime to $2n-1$ so that the union $G_i \cup G_j$ will be a Hamiltonian circuit of K_{2n}. This contradiction completes the proof of Theorem 3.1.

The proof of Theorem 3.1 suggests the following corollary.

COROLLARY 3.2. *An $STS(2v+1)$ containing a unique $STS(v)$ exists if and only if $v \geqslant 7$.*

Proof. Obviously the condition $v \geqslant 7$ is necessary. The sufficiency follows from part II of the proof of Theorem 3.1 since no assumption about $(S, \mathscr{B})$ other than that it is an $STS(v)$ is used there.

COROLLARY 3.3. $R(2v+1) \geqslant (v-2)!\, R(v)$.

Proof. Denote by $D(S, \mathscr{B})$ the set of pairwise distinct $STS(v)$ on S isomorphic to $(S, \mathscr{B})$, and by $\mathscr{R}_{2v+1}(S, \mathscr{B})$ the set of all isomorphism classes of $STS(2v+1)$ constructed as in Theorem 3.1 containing $(S, \mathscr{B})$ as its (unique) $STS(v)$, and let $\Gamma(GK_{2n})$ be the automorphism group of the 1-factorization GK_{2n}. Then we evidently have

$$| \mathscr{R}_{2v+1}(S, \mathscr{B})| = \frac{| \mathscr{D}(S, \mathscr{B})|}{| \Gamma(GK_{2n})|}.$$

If $(S, \mathscr{B})$ is an AF STS then $| \mathscr{D}(S, \mathscr{B})| = v!$. On the other hand, as mentioned earlier, $| \Gamma(GK_{2n})| \leqslant v(v-1)$. If $(S, \mathscr{B}_1)$ and $(S, \mathscr{B}_2)$ are two nonisomorphic AF $STS(v)$ then $(S^*, \mathscr{B}_1^*) \in R_{2v+1}(S, \mathscr{B}_1)$ and $(S^*, \mathscr{B}_2^*) \in \mathscr{R}_{2v+1}(S, \mathscr{B}_2)$ are also nonisomorphic, and the corollary follows.

In order to prove the next theorem we need the following lemma.

LEMMA 3.4. *There exists an $STS(49)$ containing a unique $STS(21)$.*

Proof. Denote $S_i = \{j_i \mid j = 0, 1,..., 20\}$, $i = 1, 2$ and

$$X = \{\infty_i \mid i = 0, 1,..., 6\}.$$

Let $(S_1, \mathscr{B})$ be an $STS(21)$. Put $S^* = S_1 \cup S_2 \cup X$, $\mathscr{B}^* = \mathscr{B} \cup \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ where

$\mathscr{B}_1 = \{\{\infty_i, \infty_{i+1}, \infty_{i+3}\} \mid i = 1, 2,..., 7\}$

$\mathscr{B}_2 = \{\{j_2, (j+1)_2, (j+3)_2\} \mid j = 1, 2,..., 21\}$

$$\begin{aligned}\mathscr{B}_3 = \{&\{\infty_0, j_1, j_2\}, \{\infty_1, j_1, (j+1)_2\}, \{\infty_2, j_1, (j+2)_2\},\\ &\{\infty_3, j_1, (j+3)_2\}, \{\infty_4, j_1, (j+4)_2\}, \{\infty_5, j_1, (j+5)_2\},\\ &\{\infty_6, j_1, (j+6)_2\}, \{j_1, (j+7)_2, (j+16)_2\},\\ &\{j_1, (j+8)_2, (j+15)_2\}, \{j_1, (j+9)_2, (j+14)_2\},\\ &\{j_1, (j+10)_2, (j+20)_2\}, \{j_1, (j+11)_2, (j+19)_2\},\\ &\{j_1, (j+12)_2, (j+18)_2\}, \{j_1, (j+13)_2, (j+17)_2\} \mid j = 1, 2,..., 21\}\end{aligned}$$

with subscripts in $\mathscr{B}_1$ and numbers in $\mathscr{B}_2$, $\mathscr{B}_3$ reduced whenever necessary modulo 7 and modulo 21, respectively. Then $(S^*, \mathscr{B}^*)$ can be seen to be an $STS(49)$. Assume that $(S^*, \mathscr{B}^*)$ contains $(S', \mathscr{B}')$ as an $STS(21)$, with $S' \neq S_1$. Then, since $(S_1 \cap S', \mathscr{B} \cap \mathscr{B}')$ must also be a subsystem of $(S^*, \mathscr{B}^*)$, we must have $| S_1 \cap S' | = 0, 1, 3, 7$ or 9. The first three cases are evidently impossible. If we had $| S_1 \cap S' | = 7$ then we would have $| S' \cap X | = 7$ and consequently $| S' \cap S_2 | = 7$. But then $(S' \cap S_2, \bar{\mathscr{B}})$ would also have to be an $STS(7)$—with some $\bar{\mathscr{B}} \subseteq \mathscr{B}^*$—which obviously cannot be the case. Thus the only possibility left is $| S' \cap S_1 | = 9$ which implies $| S' \cap S_2 | = 9$, $| S' \cap X | = 3$, and the three elements of $S' \cap X$ must from a triple of $\mathscr{B}^*$. In $S' \cap S_2$ there must be at least two elements, say j_2, k_2, with $j - k = 1$. Without loss of generality we may assume these two elements to be 1_2 and 2_2. Let now $S' \cap X = \{\infty_i, \infty_{i+1}, \infty_{i+3}\}$. We have thus $1_2, 2_2, \infty_i, \infty_{i+1}, \infty_{i+3} \in S'$. But $1_2, \infty_{i+1} \in S'$ implies $(-i)_1 \in S'$ while $\infty_i, (-i)_1 \in S'$ implies $0_2 \in S'$. Thus $0_2, 1_2, 2_2 \in S'$ which obviously implies that every $j_2 \in S'$, $j = 0, 1,..., 20$ which is a contradiction. Thus $(S_1, \mathscr{B})$ is the unique $STS(21)$ contained in our $(S^*, \mathscr{B}^*)$.

THEOREM 3.5. *If there exists an AF $STS(v)$ then there exists an AF $STS(2v + 7)$.*

Proof. Let $(S, \mathscr{B})$ be an AF $STS(v)$, $S = \{a_1, a_2, ..., a_v\}$, let $U = \{b_1, b_2, ..., b_v\}$, $X = \{\infty_i \mid i = 1, 2,..., 7\}$, $(X, \mathscr{D})$ an $STS(7)$. Put $\frac{1}{2}(v - 1) = s$, and let $L = \{(p_r, q_r) \mid q_r - p_r = r, r = 1, 2,..., s\}$ be an (A, s)-system or (B, s)-system according to whether $s \equiv 0$ or $1 \pmod 4$ or $s \equiv 2$ or $3 \pmod 4$. Denote further $Y = U - W$ where $W = \{b_i \mid i = p_r$ or q_r, $r = 4, 5,..., s$, $(p_r, q_r) \in L\}$. Obviously $| Y | = 7$. Now let $Y = \{b_{j_i} \mid i = 1, 2,..., 7\}$. Put now $S^* = S \cup U \cup X$, $\mathscr{B}^* = \mathscr{B} \cup \mathscr{D} \cup \mathscr{E} \cup \mathscr{F} \cup \mathscr{G}$ where

$$\mathscr{E} = \{\{\infty_i, a_k, b_{j_i+k-1}\} \mid i = 1, 2,..., 7; k = 1, 2,..., v\},$$

$$\mathscr{F} = \{\{a_k, b_{p_r+k-1}, b_{q_r+k-1}\} \mid k = 1, 2,..., v; r = 4, 5,..., s; (p_r, q_r) \in L\},$$

and

$$\mathscr{G} = \{\{b_i, b_{i+1}, b_{i+3}\} \mid i = 1, 2,..., v\}$$

with subscripts reduced modulo v whenever necessary. It is readily verified that $(S^*, \mathscr{B}^*)$ is an $STS(2v + 7)$ [13]. Let us show that $(S^*, \mathscr{B}^*)$ has no non-trivial automorphisms.

I. Assume first that there is a nontrivial automorphism $\sigma \in \Gamma(S^*, \mathscr{B}^*)$ which maps $(S, \mathscr{B})$ onto itself. Then σ must map $\mathscr{D} \cup \mathscr{G}$ onto itself. Actually, since $v \geqslant 15$, σ must map $\mathscr{D}$ onto itself and $\mathscr{G}$ onto itself because any element of X is in an $STS(7)$ while no element of U is in any $STS(7)$ on any $Z \subseteq X \cup U$. On the other hand, since by our assumption $(S, \mathscr{B})$ is an AF $STS(v)$, we have $a_i\sigma = a_i$ for each element of S and therefore the set of pairs $E_k \cup F_k$ where

$$E_k = \{(\infty_i, b_{j_i+k-1}) \mid i = 1, 2,..., 7\}$$

and

$$F_k = \{(b_{p_r+k-1}, b_{q_r+k-1}) \mid r = 4, 5,..., s; (p_r, q_r) \in L\}$$

must be fixed under σ for each $k = 1, 2,..., v$. One can see immediately that if $v \geqslant 15$ then σ would map the set $\mathscr{G}$ onto itself only if $b_i\sigma = b_{i+x}$ for each $i = 1, 2,..., v$ and for some (fixed) x. But then σ obviously cannot fix the set $E_k \cup F_k$.

II. Assume now that σ is a non-trivial automorphism of $(S^*, \mathscr{B}^*)$ which maps $(S, \mathscr{B})$ onto $(S', \mathscr{B}')$ where $(S', \mathscr{B}')$ is another $STS(v)$. Let us recall that $| S \cap S' | \leqslant \frac{1}{2}(v - 1)$. Consider the following four cases:

(i) $X \cap S' = \varnothing$. This forces $| S \cap S' | = \frac{1}{2}(v - 1)$, $| S' \backslash S | = \frac{1}{2}(v + 1)$, $S' \backslash S = \bar{S} \subseteq U$. In this case $(S', \mathscr{B}')$ cannot contain triples with all three elements belonging to $\bar{S}$ (since $(S \cap S', \mathscr{B} \cap \mathscr{B}')$ is a subsystem of $(S, \mathscr{B})$ with $2 | S \cap S' | + 1 = | S |$). On the other hand, any two elements b_x and b_y of U with $| x - y | \equiv 1, 2$ or $3 \pmod v$ are contained in a triple whose third element also belongs to U. But among the $\frac{1}{2}(v + 1)$ elements of $\bar{S}$ there must be a pair b_x, b_y with $x - y = 1, 2$ or 3 which provides the required contradiction.

(ii) $| X \cap S' | = 1$. This forces $| S \cap S' | = \frac{1}{2}(v - 1)$, $| S' \cap U | = \frac{1}{2}(v - 1)$. Again, $(S', \mathscr{B}')$ cannot contain triples with all three elements belonging to $S' \backslash S$. However, among the $\frac{1}{2}(v - 1)$ elements of $S' \cap U$ there still must be a pair b_x, b_y with $x - y = 1, 2$, or 3 generating a triple with all three elements belonging to $S' \cap U$ which is a contradiction.

(iii) $| X \cap S' | = 3$. This forces $| S \cap S' | = \frac{1}{2}(v - 3)$, $| S' \cap U | = \frac{1}{2}(v - 3)$. If one denotes by $\lambda(v)$ the maximum number of elements of U with no three consecutive subscripts (mod v) then one can easily see (say, by exhausting all possible cases) that $\lambda(v) \leqslant 3v/7$. If $S' \cap U$ contains three elements of U with consecutive subscripts (mod v) then $S' \cap U = U$. Thus we must have $\lambda(v) \geqslant \frac{1}{2}(v-3)$; i.e., $v \leqslant 21$. Consequently, if $v > 21$ we have obtained a contradiction. The case $v = 21$ is handled separately by Lemma 3.4. This is the only remaining case to be considered since we have $v \geqslant 15$ and in addition we must have $v \equiv 9 \pmod{12}$ (in order to have $\frac{1}{2}(v-3) \equiv 1$ or $3 \pmod 6$)).

(iv) $|X \cap S'| = 7$. In this case we may assume without loss of generality that $\infty_1, \infty_2, a_1, a_2, b_{j_1}, b_{j_2}$ all belong to S', and $j_2 = j_1 + 1$. (Recall that to obtain Y we "omitted" from our (A, s)-system or (B, s)-system among others the pair (p_1, q_1).) But the third element of the triple containing ∞_2 and a_2 is by our construction b_{j_2+1}. Thus our assumption implies $S' \supseteq U$ which is a contradiction. This completes the proof of Theorem 3.5.

COROLLARY 3.6. *There exists an $STS(2v+7)$ containing a unique $STS(v)$ whenever $v \geqslant 13$.*

Proof. The corollary follows from Lemma 3.4 and the proof of Theorem 3.5.

COROLLARY 3.7. $R(2v+7) \geqslant (v-1)!\, R(v)/168$.

Proof. Let $(S^*, \mathscr{B}^*)$ be as in Theorem 3.5 and consider the pair $(H, \mathscr{W})$ where

$$H = U \cup X, \qquad \mathscr{W} = \mathscr{D} \cup \mathscr{G} \cup \bigcup_{k=1}^{v} (E_k \cup F_k).$$

Thus $(H, \mathscr{W})$ is the configuration obtained from $(S^*, \mathscr{B}^*)$ by omitting from S^* (and $\mathscr{B}^*$) all the elements of S. (Actually, $(H, \mathscr{W})$ is a pairwise balanced design [7] with blocks of size two and three.) Let $\Gamma(H, \mathscr{W})$ be the automorphism group of $(H, \mathscr{W})$ (i.e., the set of mappings of H onto itself preserving $\mathscr{W}$ and, of course, preserving the sets $E_k \cup F_k$). One has

$$R(2v+7) \geqslant \frac{v!\, R(v)}{|\Gamma(H, \mathscr{W})|}.$$

As already mentioned in the proof of Theorem 3.5, $\mathscr{G}$ as a set of triples has to be mapped onto itself by any $\sigma \in \Gamma(H, \mathscr{W})$ and therefore we must have $b_i\sigma = b_{i+x}$ for each $i = 1, 2, \ldots, v$ and for some $x \in \{1, 2, \ldots, v\}$. Thus $|\Gamma(H, \mathscr{W})| \leqslant v\, |\Gamma(X, \mathscr{D})|$. To complete the proof of the corollary it is now sufficient to recall the well-known fact that the automorphism group of any $STS(7)$ has order $7 \cdot 6 \cdot 4 = 168$.

LEMMA 3.8. *There exists an AF $STS(33)$.*

Proof. Let $S_i = \{j_i \mid j = 0, 1,..., 14\}$, $i = 1, 2$, $X = \{\infty_1, \infty_2, \infty_3\}$, and $S = S_1 \cup S_2 \cup X$. Let $(S_1, \mathscr{B}(15))$ be any $AF\ STS(15)$ (which exists by [6]). Consider the following set of triples $\mathscr{B}$: $\mathscr{B} = \mathscr{B}(15) \cup \{\infty_1, \infty_2, \infty_3\} \cup \mathscr{C} \cup \mathscr{D} \cup \mathscr{E}$, where

$$\mathscr{C} = \{\{\infty_1, j_1, j_2\}, \{\infty_2, j_1, (j+1)_2\}, \{\infty_3, j_1, (j+3)_2\} \mid j = 1, 2,..., 15\},$$

$$\begin{aligned}\mathscr{D} = \{&\{j_1, (j+2)_2, (j+4)_2\}, \{j_1, (j+5)_2, (j+8)_2\},\\ &\{j_1, (j+6)_2, (j+12)_2\}, \{j_1, (j+7)_2, (j+14)_2\},\\ &\{j_1, (j+9)_2, (j+13)_2\}, \{j_1, (j+10)_2, (j+11)_2\} \mid j = 1, 2,..., 15\},\end{aligned}$$

$$\mathscr{E} = \{\{j_2, (j+5)_2, (j+10)_2\} \mid j = 1, 2, 3, 4, 5\},$$

the numbers in $\mathscr{C}$, $\mathscr{D}$, $\mathscr{E}$ reduced modulo 15 whenever necessary. It is verified directly that $(S, \mathscr{B})$ is an $STS(33)$.

If one assumes that there is a non-trivial automorphism σ of $(S, \mathscr{B})$ which maps $(S_1, \mathscr{B}(15))$ onto itself then by repeating essentially the corresponding part of the proof of Theorem 3.5 (except it is easier here since one deals with a particular STS) one is lead to a contradiction. On the other hand, our $(S, \mathscr{B})$ contains no subsystem of order 15 other than $(S_1, \mathscr{B}(15))$. For if $(S', \mathscr{B}')$ were such an $STS(15)$ then with $\bar{\mathscr{B}} = \mathscr{B}' \backslash \mathscr{B}(15)$ we would have $|\bar{\mathscr{B}}| \geqslant 8$, and consequently $|\bar{\mathscr{B}}| = 8$ which would imply that there can be no triple consisting of elements of $\bar{\mathscr{B}}$ only. However $\bar{\mathscr{B}}$ can contain at most 5 elements of S_2 (not forming a triple) and therefore $\bar{\mathscr{B}}$ has to contain ∞_1, ∞_2, ∞_3. But they form a triple giving a contradiction which proves the lemma.

In the following three lemmas we use the notion of a T-table of an STS (introduced by Cummings [3], [16] and described in detail and generalized in [11]) to verify that our STS is automorphism free. To make the reading of the present paper independent of [11] let us describe briefly this useful device. If $(S, \mathscr{B})$ is an $STS(v)$ then for any two elements $x, y \in S$ one can construct a graph G_{xy} with $V(G_{xy}) = S - \{x, y, z\}$ (where $\{x, y, z\} \in \mathscr{B}$) and $E(G_{xy}) = \{[a, b] \mid \{a, b, x\} \in \mathscr{B}$ or $\{a, b, y\} \in \mathscr{B}\}$. From this definition and from the properties of STS it follows that G_{xy} is a quadratic graph with all of its components being even cycles of length s, $4 \leqslant s \leqslant v - 3$. Thus to any two elements $x, y \in S$ corresponds a partition of $v - 3$ into even parts not less than 4. This partition is called the *type of interlacing* of x and y. If $T_1, T_2,..., T_q$ are all such partitions then to every $x \in S$ one can assign a vector $t = (t_1, t_2,..., t_q)$ called the *vector-index* of x where t_i is the number of elements of S having with x the type of interlacing T_i. Denoting by $N(t)$ the number of elements of S having t as its vector-index, one obtains the T-table of the given $STS(v)$:

t_1^1	t_2^1	$\cdots$	t_q^1	$N(t^1)$
t_1^2	t_2^2	$\cdots$	t_q^2	$N(t^2)$
$\cdots$				$\cdots$
t_1^r	t_2^r	$\cdots$	t_q^r	$N(t^r)$

where, of course, $\sum_{i=1}^{q} t_i^j = v - 3$, $\sum_{i=1}^{r} N(t^i) = v$.

Two $STS(v)$ with different T-tables are non-isomorphic. Moreover two elements of $(S, \mathscr{B})$ having different vector-indices belong to different orbits under $\Gamma(S, \mathscr{B})$. This, together with the well known fact that any automorphism of $(S, \mathscr{B})$ fixing at least half the number of elements of S necessarily fixes all elements of S will enable us to prove certain STS to be automorphism free. The method of T-tables besides enabling us to prove such a result, has also the advantage that the verification can be presented in a neat form so that the reader can see it "at one glance". (This remark, however, should not imply that the T-table itself can be constructed easily: in order to obtain the T-table of an $STS(v)$ one has to construct, in general, $\binom{v}{2}$ graphs G_{xy} to determine the types of interlacing.)

LEMMA 3.9. *There exists an AF* $STS(27)$.

Proof. Let $S_i = \{j_i \mid j = 0, 1,..., 12\}$, $i = 1, 2$, $S = S_1 \cup S_2 \cup \{\infty\}$. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ where

$$\begin{aligned}
\mathscr{B}_1 = \{&\{\infty, j_1, j_2\}, \{(j+1)_1, (j+12)_1, j_2\}, \{(j+2)_1, (j+11)_1, j_2\},\\
&\{(j+3)_1, (j+10)_1, j_2\}, \{(j+4)_1, (j+9)_1, j_2\},\\
&\{(j+5)_1, (j+8)_1, j_2\}, \{(j+6)_1, (j+7)_1, j_2\} \mid j = 1, 2,..., 13\},\\
\mathscr{B}_2 = \{&\{0_2, 1_2, 2_2\}, \{0_2, 3_2, 4_2\}, \{0_2, 5_2, 6_2\}, \{0_2, 7_2, 8_2\}, \{0_2, 9_2, 10_2\},\\
&\{0_2, 11_2, 12_2\}, \{1_2, 3_2, 5_2\}, \{1_2, 4_2, 6_2\}, \{1_2, 7_2, 9_2\}, \{1_2, 8_2, 11_2\},\\
&\{1_2, 10_2, 12_2\}, \{2_2, 3_2, 7_2\}, \{2_2, 4_2, 11_2\}, \{2_2, 5_2, 12_2\}, \{2_2, 8_2, 9_2\},\\
&\{2_2, 6_2, 10_2\}, \{3_2, 6_2, 8_2\}, \{3_2, 9_2, 12_2\}, \{3_2, 10_2, 11_2\}, \{4_2, 7_2, 10_2\},\\
&\{4_2, 5_2, 9_2\}, \{4_2, 8_2, 12_2\}, \{5_2, 7_2, 11_2\}, \{5_2, 8_2, 10_2\}, \{6_2, 7_2, 12_2\},\\
&\{6_2, 9_2, 11_2\}\},
\end{aligned}$$

with the numbers in $\mathscr{B}_1$ reduced modulo 13. Observe that $(S_2, \mathscr{B}_2)$ is an $STS(13)$ isomorphic to the "non-cyclic" $STS(13)$ (cf. [7], p. 237, the "second solution"). The T-table (Table 1) shows that our system $(S, \mathscr{B})$ is automorphism free. The notations for the partitions are: $T_1 = (24)$, $T_2 = (20, 4)$, $T_3 = (16, 8)$, $T_4 = (16, 4, 4)$, $T_5 = (14, 10)$, $T_6 = (14, 6, 4)$, $T_7 = (12, 12)$, $T_8 = (12, 8, 4)$, $T_9 = (8, 8, 8)$, $T_{10} = (8, 8, 4, 4)$ (only those partitions are shown which actually occur as types of interlacing).

TABLE 1

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	$N(t_i)$
20	3	1	1	0	0	1	0	0	0	1
19	4	1	1	0	0	0	0	0	1	1
19	3	2	0	0	0	1	1	0	0	1
19	3	1	1	0	0	1	0	1	0	1
18	2	2	1	0	0	2	0	0	1	1
18	2	1	1	0	0	1	1	1	1	1
18	2	0	0	0	0	1	3	2	0	1
18	1	0	3	0	0	0	3	1	0	1
17	2	3	0	0	0	1	2	1	0	1
16	5	3	1	0	0	0	0	0	1	1
16	3	1	3	0	0	1	1	0	1	1
16	1	5	1	0	0	0	2	0	1	1
15	4	0	4	0	0	0	2	0	1	1
15	3	0	2	0	0	2	3	1	0	1
7	2	2	3	8	4	0	0	0	0	1
6	5	1	1	8	4	1	0	0	0	1
6	4	1	3	8	4	0	0	0	0	1
6	1	0	2	8	4	2	1	1	1	1
5	4	3	1	8	4	0	1	0	0	1
5	3	2	0	9	3	0	2	1	1	1
5	3	0	1	8	4	2	3	0	0	1
5	1	2	2	9	3	2	2	0	0	1
4	6	0	0	9	3	2	1	0	1	1
4	3	3	3	9	3	0	1	0	0	1
4	2	2	3	8	4	2	1	0	0	1
3	2	2	0	8	4	0	4	1	2	1
2	2	2	0	8	4	0	2	4	2	1

LEMMA 3.10. *There exists an AF $STS(25)$.*

Proof. Let $S_i = \{j_i \mid j = 0, 1,..., 8\}$, $i = 1, 2$, $X = \{\infty_i \mid i = 0, 1,..., 6\}$, and $S = S_1 \cup S_2 \cup X$. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3 \cup \mathscr{B}_4$ where

$$\mathscr{B}_1 = \{\{\infty_i, \infty_{i+1}, \infty_{i+3}\} \mid i = 1, 2,..., 7\},$$
$$\mathscr{B}_2 = \{\{0_1, 1_1, 2_1\}, \{0_1, 3_1, 6_1\}, \{0_1, 4_1, 7_1\}, \{0_1, 5_1, 8_1\}, \{3_1, 4_1, 5_1\}, \{1_1, 5_1, 7_1\}, \{1_1, 3_1, 8_1\}, \{1_1, 4_1, 6_1\}, \{6_1, 7_1, 8_1\}, \{2_1, 4_1, 8_1\}, \{2_1, 5_1, 6_1\}, \{2_1, 3_1, 7_1\}\},$$
$$\mathscr{B}_3 = \{\{j_2, (j+1)_2, (j+3)_2\} \mid j = 1, 2,..., 9\},$$

and

$$\mathscr{B}_4 = \{\{\infty_0, j_1, j_2\}, \{\infty_1, j_1, (j+1)_2\}, \{\infty_2, j_1, (j+2)_2\}, \{\infty_3, j_1, (j+3)_2\}, \{\infty_4, j_1, (j+5)_2\}, \{\infty_5, j_1, (j+6)_2\}, \{\infty_6, j_1, (j+7)_2\}, \{j_1, (j+4)_2, (j+8)_2\} \mid j = 1, 2,..., 9\}.$$

One verifies in a straightforward manner that $(S, \mathscr{B})$ is an $STS(25)$. The T-table presented below (Table 2) shows that the only automorphism of $(S, \mathscr{B})$ is the identity. The partitions are denoted as follows (again only those partitions are shown which actually occur): $T_1 = (22)$, $T_2 = (18, 4)$, $T_3 = (16, 6)$, $T_4 = (14, 8)$, $T_5 = (12, 10)$, $T_6 = (12, 6, 4)$, $T_7 = (10, 8, 4)$, $T_8 = (10, 6, 6)$, $T_9 = (8, 8, 6)$, $T_{10} = (6, 6, 6, 4)$.

LEMMA 3.11. *There exists an AF $STS(21)$.*

Proof. Let $S_i = \{j_i \mid j = 0, 1,..., 8\}$, $i = 1, 2$, and $S = S_1 \cup S_2 \cup \{\infty_1, \infty_2, \infty_3\}$. Let $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ where

$$\mathscr{B}_1 = \{\{\infty_1, \infty_2, \infty_3\}, \{0_2, 3_2, 6_2\}, \{1_2, 4_2, 7_2\}, \{2_2, 5_2, 8_2\}\},$$

$\mathscr{B}_2$ is the same as in Lemma 3.10, and

$$\begin{aligned}\mathscr{B}_3 = \{&\{\infty_1, j_1, j_2\}, \{\infty_2, j_1, (j+1)_2\}, \{\infty_3, j_1, (j+2)_2\},\\ &\{j_1, (j+3)_2, (j+5)_2\}, \{j_1, (j+4)_2, (j+8)_2\},\\ &\{j_1, (j+6)_2, (j+7)_2\} \mid j = 1, 2,..., 9\}\end{aligned}$$

TABLE 2

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	$N(t_i)$
15	6	1	0	0	1	0	0	0	1	1
15	5	0	0	2	0	0	0	0	2	1
14	5	1	0	2	0	0	0	0	2	1
13	8	1	0	1	0	0	0	0	1	1
7	15	0	0	1	0	0	0	0	1	1
6	5	1	9	2	0	0	0	0	1	1
6	5	1	0	10	0	0	0	0	2	1
14	1	0	4	3	0	0	0	2	0	1
14	0	0	3	4	0	1	0	2	0	1
13	2	1	2	4	0	0	0	2	0	1
13	2	0	3	3	0	1	0	2	0	1
13	1	1	3	4	0	0	0	2	0	1
13	1	0	2	5	0	1	0	2	0	1
12	4	0	3	3	0	0	0	2	0	1
12	2	0	4	4	0	0	0	2	0	1
12	2	0	3	5	0	0	0	2	0	1
11	2	6	1	1	0	1	2	0	0	1
10	3	6	1	2	0	0	2	0	0	1
10	3	6	1	1	0	1	2	0	0	1
10	2	6	1	3	0	0	2	0	0	2
9	4	6	3	0	0	0	2	0	0	1
9	2	9	0	1	0	1	2	0	0	1
8	0	9	3	2	0	0	2	0	0	1
7	2	7	1	4	1	0	2	0	0	1

with numbers in $\mathscr{B}_3$ reduced modulo 9. One easily verifies $(S, \mathscr{B})$ to be an $STS(21)$. The T-table below (Table 3) shows that $(S, \mathscr{B})$ is automorphism free. The partitions are denoted as follows: $T_1 = (18)$, $T_2 = (14, 4)$, $T_3 = (12, 6)$, $T_4 = (10, 8)$, $T_5 = (10, 4, 4)$, $T_6 = (8, 6, 4)$, $T_7 = (6, 6, 6)$.

THEOREM 3.12. *An AF $STS(v)$ exists if and only if $v \geqslant 15$.*

Proof. As has already been mentioned in Section 3, there is no $AF\ STS(v)$ for $v \leqslant 13$. It is also well-known that there exists an $AF\ STS(15)$ (actually, there are exactly 36 nonisomorphic $AF\ STS(15)$ [6, 16]) and an $AF\ STS(19)$ (see [4] and also [10] for an independent verification that the system in [4] is an $AF\ STS$). By Lemmas 3.8–3.11 there exists an $AF\ STS(v)$ for $v = 21$, 25, 27, 33 and by Theorem 3.1 there exists an $AF\ STS(31)$. Assume therefore $v \geqslant 37$, and assume that for all admissible $v' < v$ $(v' \geqslant 15)$ there exists an $AF\ STS(v')$. If $v \equiv 3$ or 7 (mod 12) then $\frac{1}{2}(v - 1) \equiv 1$ or 3 (mod 6) and $\frac{1}{2}(v - 1) \geqslant 19$. Therefore there is an $AF\ STS(\frac{1}{2}(v - 1))$ and by Theorem 3.1 there is an $AF\ STS(v)$. If $v \equiv 1$ or 9 (mod 12) then $\frac{1}{2}(v - 7) \equiv 1$ or 3 (mod 6), $\frac{1}{2}(v - 7) \geqslant 15$. Therefore there is an $AF\ STS(\frac{1}{2}(v - 7))$ and by Theorem 3.5 there is an $AF\ STS(v)$.

THEOREM 3.13. $\lim_{v \to \infty} R(v) = \infty$.

Proof. The statement follows from Corollary 3.3 and Corollary 3.7.

TABLE 3

T_1	T_2	T_3	T_4	T_5	T_6	T_7	$N(t_i)$
14	3	2	0	0	1	0	1
12	1	2	3	1	1	0	1
6	1	11	1	0	1	0	1
13	3	2	0	0	2	0	1
13	3	1	1	0	2	0	1
13	2	2	0	1	2	0	1
12	4	1	1	0	2	0	1
12	3	3	0	0	2	0	1
11	4	1	2	0	2	0	1
10	5	1	1	1	2	0	1
9	5	2	1	1	2	0	1
8	5	2	2	0	3	0	1
7	3	4	1	0	3	2	1
7	2	5	1	0	3	2	1
6	3	6	0	1	2	2	1
6	2	7	1	0	2	2	1
6	2	6	2	0	2	2	2
6	2	4	3	0	3	2	1
6	1	5	1	2	3	2	1
5	4	5	1	1	2	2	1

4. $R(v)$ From $v = 15$ to 103

The following table (Table 4) gives the best results to date for $R(v)$ from $v = 15$ to 103. The bounds listed without explanation were obtained by techniques other than those developed in this paper and will be the subject of a subsequent paper by the authors.

TABLE 4

	$R(v)$			$R(v)$	
15	36	[6, 16]	61	$\geqslant \frac{1}{168} \cdot 26!$	(3.7)
19	$\geqslant 1$	[4, 11]	63	$\geqslant 36 \cdot 13!\, 29!$	(3.3)
21	$\geqslant 1$	(3.11)	67	$\geqslant 36 \cdot 31!$	(3.3)
25	$\geqslant 1$	(3.10)	69	$\geqslant \frac{3}{14} \cdot 13!\, 30!$	(3.7)
27	$\geqslant 1$	(3.9)	73	$\geqslant \frac{1}{14} \cdot 32!$	(3.7)
31	$\geqslant 36 \cdot 13!$	(3.3)	75	$\geqslant 22!\, 21! \cdots 2 \cdot 1$	
33	$\geqslant 1$	(3.8)	79	$\geqslant 17!\, 37!$	(3.3)
37	$\geqslant 3 \cdot 13!$	(3.7)	81	$\geqslant \frac{9}{14} \cdot 13!\, 35!$	(3.7)
39	$\geqslant 12!\, 11! \cdots 2 \cdot 1$		85	$\geqslant \frac{1}{168} \cdot 17!\, 38!$	(3.7)
43	$\geqslant 14!\, 13! \cdots 2 \cdot 1$		87	$\geqslant 25!\, 24! \cdots 2 \cdot 1$	
45	$\geqslant 15!\, 14! \cdots 2 \cdot 1$		91	$\geqslant \frac{3}{28} \cdot 17!\, 43!$	(3.3)
49	$\geqslant \frac{1}{168} \cdot 19!$	(3.7)	93	$\geqslant 28!\, 27! \cdots 2 \cdot 1$	
51	$\geqslant 13!\, 12! \cdots 2 \cdot 1$		97	$\geqslant 29!\, 28! \cdots 2 \cdot 1$	
55	$\geqslant 15!\, 14! \cdots 2 \cdot 1$		99	$\geqslant 30!\, 31! \cdots 2 \cdot 1$	
57	$\geqslant 16!\, 15! \cdots 2 \cdot 1$		103	$\geqslant 23!\, 49!$	(3.3)

References

1. B. A. Anderson, Finite topologies and Hamiltonian paths, *J. Combinat. Theory*, **14** (1973), 87–93.
2. B. A. Anderson, Symmetry groups of perfect 1-factorizations on some K_{2n}, to appear.
3. L. D. Cummings, On a method of comparison for triple-systems, *Trans. Amer. Math. Soc.* **15** (1914), 311–327.
4. J. Doyen, Sur la structure de certains systèmes triples de Steiner, *Math. Z.* 111 (1969), 289–300.
5. J. Doyen, Recent results on Steiner triple systems, Finite Geometric Structures and their Applications, Proc. Confer. Bressanone, June 1972, Edizioni Cremonese, Roma 1973, pp. 203–210.
6. R. A. Fisher, An examination of the different possible solutions of a problem in incomplete blocks, *Ann. Eugenics*, **10** (1940), 52–75.
7. M. Hall, Jr., "Combinatorial Theory," Blaisdell Waltham MA 1967.
8. F. Harary, "Graph Theory" Addison-Wesley Reading MA 1969.
9. A. Kotzig, Hamilton graphs and Hamilton circuits *in* "Theory of Graphs and its Applications pp. 63–82" Proc. Symposium Smolenice 1963 Academic Press, New York, 1964.

10. A. J. PETRENYUK, Non-isomorphic headed Steiner triple systems and systems of pair-groups (in Russian), *Kibernetika* **7** (1971), 93–102.
11. A. J. PETRENYUK, Criteria of non-isomorphism of Steiner triple systems (in Russian), *Ukrain Mat. Ž.* **24** (1972), 772–780.
12. A. ROSA, Remark on cyclic Steiner triple systems (in Slovak), *Mat. Fyz. Časopis*, **16** (1966), 285–290.
13. A. ROSA, Steiner triple systems and their chromatic number, *Acta F. R. N. Univ. Comen. Mathematica*, **24** (1970), 159–174.
14. D. P. ROSELLE, Distributions of integers into s-tuples with given differences, pp. 31–42, Proc. of the Manitoba Conference on Numberical Mathematics, Winnipeg, 1971.
15. W. D. WALLIS, On one-factorizations of complete graphs, *J. Austral. Math. Soc.* **16** (1973), 167–171.
16. H. S. WHITE, F. N. COLE, L. D. CUMMINGS, Complete classification of the triad systems on fifteen elements, *Mem. Nat. Acad. Sci. U.S.A.* **14**, 2nd memoir (1919), 1–89.